LINEAR PROGRAMMING.

→ A set S is said to be <u>convex</u> if for any elements $X_1, X_2 \in S$, $\lambda X_2 + (1-\lambda) X_1 \in S$ for $0 \le \lambda \le 1$.

→ Let S be a convex set. A point $x \in S$ is an <u>extreme pt.</u> of S if and only if there donot exist points x_1, x_2 $(x_1 \neq x_2)$ in the set S such that $x = \lambda x_1 + (1-\lambda) x_2$: $0 < \lambda < 1$.

→ A <u>hyperplane</u> in E^n is defined to be a set 'S' of points $S = \{x \in E^n / c_1 x_1 + c_2 x_2 + \dots + c_n x_n = d\}$, i.e. $S = \{x \in E^n / CX = d\}$. It divides E^n into 3 mutually exclusive and exhaustive regions, $S_1 = \{x : CX < d\}$, $S_2 = \{x : CX = d\}$, $S_3 = \{x : CX > d\}$. S_1 and S_3 are called open <u>halfspaces</u>. $S_4 = \{x : CX \le d\}$ and $S_5 = \{x : CX \ge d\}$ are called closed half spaces. Also, $S_4 \cap S_5 = S_2$.

→ <u>Convex combination</u> of pts. $x_1, x_2, \dots, x_m$ is $X = \sum_{i=1}^{m} \mu_i x_i$; $\mu_i \ge 0$ where $\sum_{i=1}^{m} \mu_i = 1$.

→ Suppose set A is not convex. Then the smallest convex set which contains A is called the <u>convex hull</u> of A. i.e. it is the intersection of all convex sets which contain A.

→ Convex hull of pts. $x_1, x_2, \dots, x_m$ is (called convex polyhedron) $S = \{X \in E^n / X = \sum_{i=1}^{m} \mu_i x_i,\ \mu_i \ge 0,\ \sum_{i=1}^{m} \mu_i = 1\}$.

→ Formulation of an LPP : <u>Step 1</u> = decision variables $x_1, x_2, \dots, x_n$; <u>Step 2</u> = formulate objective fn as a linear fn ; <u>Step 3</u> = formulate constraints ; <u>Step 4</u> = Non negativity of X.

→ The 2 techniques of solving an LPP by graphical method are (i) <u>corner point method</u> : find coordinates of each vertex of feasible region and evaluate obj fn z at each of them, (ii) <u>Iso-profit</u>/<u>Iso-cost method</u> : draw a line of the obj fn and move it around in the feasible region to find the optimal solution.

→ <u>Canonical form</u> : Maximize $Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$, subject to the constraints $a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \le b_i$: $i = 1, 2, \dots, m$, and $x_1, x_2, \dots, x_n \ge 0$.

→ <u>Standard form</u> : Maximize (or Minimize) $Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$, subject to $a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i$: $i = 1, 2, \dots, m$, and $x_1, x_2, \dots, x_n \ge 0$. Also, here $b_i \ge 0$.

→ An initial feasible solution is obtained by setting each slack and artificial variable equal to the RHS and all other variables as zero.

→ Matrix form of LPP in standard form: Max or Min $Z = C^T X$, subject to $AX = B$ and $X \geq 0$. (where X is a column vector of unknowns). Also $B \geq 0$

→ Let B be any $m \times m$ submatrix of A formed by m L.I. columns. Then solution obtained by setting $n-m$ variables not associated w/columns of B (Non basic variables) equal to zero and solving the resulting system is called a basic solution X_B. B is called the basis and its columns are the basis vectors. $X_B = B^{-1} b$.

→ Basic feasible solution (BFS) is that basic soln. which also satisfies non-ve restrictions.

→ If one or more of the basic variables is equal to zero, then the BFS is degenerate.

→ If all basic variables are +ve, the BFS is non-degenerate.

→ Total number of basic solutions $= {}^nC_m$.

→ $AX = B$ in the vector form is $x_1 A_1 + x_2 A_2 + \ldots + x_n A_n = b$: A_i is col. of A.

→ Simplex Procedure: (1) Always Maximize obj fn. and make $b_i \geq 0$; (2) express problem in standard form; (3) Find an initial BFS (IBFS); (4) Apply optimality test, i.e. if all $\Delta_j = c_j - z_j$: $z_j = \sum c_B a_{ij}$ are -ve, then IBFS is optimal. Otherwise go to next step; (5) Choose incoming variable as the as the one with max +ve c_j (if 2 are same, choose arbitrarily) and outgoing variable as the one with minimum ratio θ (≥ 0); (6) Make the incoming column as a part of basic columns using row operations; (7) Go to step (4).

→ Big M method: (1) Express the problem in standard form; (2) Assign reqd. artificial variables with very large penalty '-M'; (3) Now solve by simplex (i) if no artificial variable (a_i) in basis and optimality conditions satisfied, then OBFS is obtained, (ii) if at least one a_i in basis at zero level, then solution is degenerate OBFS (iii) if at least one a_i in basis at non-zero level, then no feasible soln. exists (aka pseudo optimal soln.)

→ Two Phase method: Phase I: (1) express LPP in standard form; (2) formulate artificial obj fn $Z^* = -a_1 - a_2 - \ldots - a_m$; (3) Now solve by simplex method (i) Max $Z^* < 0$ & at least one artificial variable in optimal basis, then no feasible soln. exists, (ii) Max $Z^* = 0$ & at least one artificial variable appears in the optimal basis at zero level, then push the artificial variable out of the basis to obtain a BFS, (iii) Max $Z^* = 0$ and no artificial variable appears in optimal basis, then BFS is obtained.

Phase II: The final simplex tableau of Phase I is taken as the initial simplex tableau of Phase II and the obj function is replaced by original obj fn.

{if $\theta \neq 0$, then choose arbitrarily.}

→ In case of tie for outgoing variable, if the equal values of θ ratios are zero, the simplex method fails. Then we use the degeneracy technique. To avoid cycling (i) divide each element in the tied rows by the +ve coeff of the key column in that row. Compare resulting ratios (from left to right) first of unit matrix and then of body matrix, column by column, (iii) The outgoing variable lies in the row which first contains the smallest algebraic ratio.

→ To construct dual problem, (1) Max in primal becomes Min in dual and vice versa, (2) $(\leq)$ in primal becomes $(\geq)$ in dual and vice versa, (3) $c_1, c_2, \dots, c_n$ of primal become $b_1, b_2, \dots, b_n$ of dual and vice versa, (4) if primal has n variables & m constraints, then dual has m variables and n constraints, (5) variables in both are non-negative.

→ Whenever there is an equality constraint, the corresponding variable becomes unrestricted and vice versa. Thus we can make it non-negative in the LPP using $x_i = x_i' - x_i''$; $x_i', x_i'' \geq 0$

→ Dual of the dual is the primal.

→ If the primal and dual have feasible solutions then both have optimal solutions and the optimal value of the primal is equal to that of dual. (Duality theorem)

→ If primal has unbounded soln. then dual is infeasible and vice versa.

→ The optimal value of primal variable is given by the coefficient of the slack variable w/ changed sign in the Δ_j row of the optimal dual simplex table & vice versa.

→ And in case of artificial variable, it is the coeff in Δ_j row after deleting M.

→ Dual Simplex method: (1) Convert the max problem into canonical form and add slack variables; (2) Find an IBS; (3) Compute $\Delta_j = c_j - z_j$, (i) iff all $\Delta_j \leq 0$ & $b_i \geq 0$, then OBFS is attained, (ii) If all $\Delta_j \leq 0$ and any $b_i < 0$, then go to step 4, (iii) if any $\Delta_j > 0$, then the method fails; (4) Mark the outgoing variable first as the one that contains most -ve b_i; (5) if all elts in this row ≥ 0, then no feasible soln.

Otherwise choose the incoming variable as the one with minimum Δ_j / a_{ij};

(6) Iterate towards OBFS. Dual Simplex: (infeasible, optimal) → (feasible, optimal)

→ Primal Simplex: (feasible, non-optimal) → (feasible optimal).

→ For a given $m \times n$ transportation problem (TP), A is $(m+n) \times mn$ matrix & its rank is $(m+n-1)$. Thus OBFS has atmost $\underline{(m+n-1)}$ +ve values.

→ There are 3 methods to obtain an IBFS for a TP: (i) North-West Corner method, (ii) Least Cost or Matrix Minima method, (iii) Vogel's Approx method.

→ In NWC, start from the upper left corner and allocate to all cells according to the supply and demand, regardless of the transportation costs.

→ In LCM/MMM, start transport with the cheapest route and use the routes in the ascending order of transportation costs.

→ In VAM, take differences b/w the min and next to min transportation costs in each row & each column, iteratively. In each iteration, choose the largest difference row/column, and allocate ~~and~~ as much as possible to the lowest cost cell. In case of tie, allocate to the cell w/lower cost among them.

→ If a closed circuit involving any basic cells can be formed, then they are in L.D positions. If no such circuit is possible, then they are in L.I positions

→ To identify the cells at zero level in a degenerate solution: such a cell can be selected which doesnot form a closed circuit with any of the other basic cells.

→ TP Procedure (Modified distribution → (MoDi/u-v method)): (1) find IBFS by NWCM or LCM or VAM; (2) assign ϵ or 0 appropriately if there is degeneracy; (3) solve $u_i + v_j = c_{ij}$ for basic cells w/ $u_i = 0$ or $v_j = 0$ for a row/col w/max no of basic cells; (4) compute $\Delta_{ij} = u_i + v_j - c_{ij}$ for all non-basic cells; (5) If all $\Delta_{ij} \leq 0$, then current BFS is optimal. Otherwise choose incoming cell w/largest $\Delta_{ij} > 0$ (if there is a tie, then go for least c_{ij}); (6) Allocate θ to chosen cell and identify a closed loop; add & subtract θ alternatively; (7) assign a max value of θ so that all variables are ≥ 0; (8) Return to step (3) until OBFS is obtained.

→ If the TP is unbalanced, then add an artificial source/destination and solve it as a balanced one. Assign the corr. costs as 0 and delete the artificial variables from the solution.

→ Assignment Problem: $\text{Min } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

s.t. $\sum_{i=1}^{m} x_{ij} = 1 \quad (j = 1, 2, \ldots, n)$

$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots, m)$

$x_{ij} = 0$ or 1
$i = 1, 2, \ldots, m$
$j = 1, 2, \ldots, n$.

→ Hungarian method: (1) subtract minimum element of each row and column from all elements of that row and column; (2) Cover all zeros with minimum no. of horizontal and vertical lines, say r. If $r = n$, optimal assignment can be made; go to step (4). If $r < n$, go to step (3); (3) Pick minimum element not covered by the lines and (i) subtract it from all uncovered elements, (ii) add to all elements at intersection of 2 covering lines, & (iii) leave all other covered elements unchanged; (4) Make an optical assignment such that each row and each column has only one zero encircled; (5) Add all corresponding c_{ij}'s to obtain minimum cost value.